

Developments in special geometry

Thomas Mohaupt¹ and Owen Vaughan^{1,2}

¹ Department of Mathematical Sciences, University of Liverpool, Peach Street, Liverpool, L69 7ZL, UK

² Department of Mathematics, University of Hamburg, Bundesstr. 55, D-20146 Hamburg, Germany

E-mail: Thomas.Mohaupt@liv.ac.uk, Owen.Vaughan@liv.ac.uk,
Owen.Vaughan@math.uni-hamburg.de

Abstract. We review the special geometry of $\mathcal{N} = 2$ supersymmetric vector and hypermultiplets with emphasis on recent developments and applications. A new formulation of the local c -map based on the Hesse potential and special real coordinates is presented. Other recent developments include the Euclidean version of special geometry, and generalizations of special geometry to non-supersymmetric theories. As applications we discuss the proof that the local r -map and c -map preserve geodesic completeness, and the construction of four- and five-dimensional static solutions through dimensional reduction over time. The shared features of the real, complex and quaternionic version of special geometry are stressed throughout.

1. Introduction

The special geometry of four-dimensional vector multiplets [1] has played a central role in studying the non-perturbative dynamics of field theories [2, 3], supergravity, and string compactifications [4, 5, 6]. It also has been central in the studies of black holes, notably in the black hole attractor mechanism [7, 8, 9, 10, 11, 12, 13, 14], and in the microscopic interpretation of black hole entropy in the context of string theory [15, 16]. Over time, various re-formulations of the original definition have been found, each with its distinguished advantages, and there has been progress in uncovering the underlying geometry [17, 18, 19, 20, 21, 22]. By dimensional lifting and reduction four-dimensional vector multiplets are related to five-dimensional vector multiplets and to three-dimensional hypermultiplets, respectively. The maps between the scalar geometries induced by dimensional reduction from 5 to 4 and from 4 to 3 dimensions are known as the r -map and the c -map, respectively [23, 24, 25, 26]. Since the corresponding geometries are closely related, we will refer to all of them as special geometries.

While this has been an active area of research for decades, there are still many open questions and further directions to pursue. One important open problem is to master the quantum corrections in the hypermultiplet sector of $\mathcal{N} = 2$ string compactifications [27, 28, 29]. Here the relation between vector and hypermultiplets induced by the c -map provides the starting point. Hypermultiplets are hard to deal with because the underlying geometry, quaternion-Kähler geometry, is more complicated and less understood than the other special Riemannian holonomy geometries on Berger's list. Another developing field is the definition and investigation of the versions of special geometry which occur in Euclidean supersymmetric theories [30, 31, 32]. This has two closely related applications: the construction of instanton solutions, and, by dimensional lifting, the construction of stationary higher-dimensional solitonic solutions, such as black holes

[33, 34, 35]. The Euclidean versions of the special geometries are systematically related to their standard counterparts through replacing complex structures by para-complex structures. This provides a framework for dealing with the analytic continuations needed to describe instanton solutions involving axionic scalars.

The approach we are taking towards special geometry combines the insights gained from the superconformal calculus and electric-magnetic duality with modern differential geometry. In this article we stress the shared features of five-dimensional and four-dimensional vector multiplets and of hypermultiplets, and we present the corresponding geometries as the real, complex and quaternionic version of the same theme. In all cases, the scalar manifolds appearing in supergravity can be obtained by ‘superconformal quotients’ from an associated superconformal theory. Conversely the scalar manifolds of the superconformal theories are real, complex and quaternionic cones (or at least ‘conical’, in a sense to be made precise later) over the scalar manifolds of the supergravity theories. We remark that tensor multiplets also seem to fit into this pattern [36], implying that there is yet another type of special geometry to be placed between the complex and the quaternionic case, since tensor multiplets have 3 real scalars. However, this geometry is not well understood, and we will not discuss it in this article.

Electric-magnetic duality occurs in four-dimensional vector multiplets and implies the invariance of the field equations under symplectic transformations. It also imprints itself by dimensional reduction on three-dimensional hypermultiplets via the c -map, giving the resulting scalar manifold the structure of a symplectic vector bundle [37, 38]. One recent observation is that symplectic covariance often can be handled better when using a formulation of special geometry in terms of special real instead of special holomorphic coordinates [39, 40, 38]. In this formulation the metric is a Hessian metric, i.e. it can be written as the second derivative of a real function, the Hesse potential. The decisive role of such Hesse potentials is a shared feature of all three types of special geometry discussed in this article.

One particularly interesting result is a new description of the local (supergravity) c -map in terms of special real coordinates and the Hesse potential [41]. In this formulation metrics obtained from the local c -map look very similar to those obtained by the rigid version. While in [38] the terms involving the three-dimensional scalars descending from four-dimensional gauge fields were expressed in terms of a real coupling matrix, we have now succeeded to relate this matrix to the Hesse potential of the four-dimensional theory, and to extend this description to all three-dimensional scalars. Since there is no natural set of special real coordinates for the scalar manifold of the four-dimensional supergravity theory which preserves the full symplectic group [42], we use the gauge equivalence with the corresponding superconformal theory, for which a description in terms of special real coordinates exists.

The formalism which we are going to review relies on the existence of a potential, from which all couplings can be derived. It does not require any non-generic global symmetries, and thus applies as well to scalar manifolds which are not homogeneous spaces (or even symmetric spaces). The advantages of the improved formulation and understanding of special geometry are demonstrated by two applications. The first is the proof that both the r -map and c -map preserve geodesic completeness (without any assumptions about non-generic isometries). This is a very useful result because it provides a method for constructing complete, but generically non-homogeneous quaternion-Kähler manifolds starting from much simpler complete real manifolds, which are fully encoded in a homogeneous cubic polynomial. While the hypermultiplet manifolds of $\mathcal{N} = 2$ string compactifications are not geodesically complete, their finite distance singularities are due to very specific non-perturbative effects, such as the appearance of additional massless modes and topological phase transitions. The results of [38] allow to start with a tree level approximation which is guaranteed to be free of unphysical singularities if the theory can be obtained from a five-dimensional theory by dimensional reduction.

The second application is the construction of extremal, but not necessarily supersymmetric,

multi-centered black hole solutions in five and four dimensions by dimensional reduction to a Euclidean theory in four and three dimensions, respectively [33, 41]. In both cases the celebrated black hole attractor equations are derived elegantly from geometrical considerations in the time-reduced theory. That black hole solutions are given in terms of harmonic functions is implied by the field equations of the reduced theory being the equations for a harmonic map from space into the scalar manifold. For BPS-type solutions, which satisfy a certain set of Bogomol'nyi type equations, one can show both by general arguments and by the introduction of suitable coordinates that the target of the harmonic map is a flat, totally geodesic, totally isotropic submanifold, and, hence, that the non-linear second order field equations reduce to decoupled Laplace equations. The use of special real coordinates and of the Hesse potential is central for directly obtaining the manifestly symplectically covariant formulation of the attractor equations.

Within our approach one becomes automatically aware that there is a natural generalization of special geometry which leads us out of the realm of supersymmetric theories. The key features underlying the above results, namely the existence of a potential encoding all couplings and the homogeneity properties implied by the relation between superconformal and Poincaré supergravity, do not depend on supersymmetry and can be generalized. In the case of real special geometry this simply amounts to allowing the potential to have an arbitrary rather than prescribed degree of homogeneity. The perservation of geodesic completeness by the r -map, the form of the five-dimensional attractor equations, and the construction of five-dimensional extremal multi-centered solutions generalize immediately [33, 38]. Moreover, once supersymmetry is not insisted on, the five dimensionally formalism can be easily be adapted to any dimension. However, in four dimensions pointlike sources can carry magnetic in addition to electric charge, and this implies additional features which still need to be analysed further. It is not straightforward to define a generalized version of special Kähler geometry, because any such generalization will imply different homogeneity properties for the electric and magnetic degrees of freedom. This case requires further study.

2. Overview of special geometries

In this article the term ‘special geometry’ refers to the geometries of vector and hypermultiplets in theories with eight real supercharges, corresponding to $\mathcal{N} = 2$ extended supersymmetry in four dimensions. Specifically, the terms ‘affine special real geometry’ and ‘projective special real geometry’ refer to the geometry of five-dimensional vector multiplets with rigid and local supersymmetry, respectively, while ‘affine special Kähler geometry’ and ‘projective special Kähler geometry’ refer to four dimensional vector multiplets. The geometries of hypermultiplets are hyper-Kähler, and quaternion-Kähler, respectively.

2.1. Hypermultiplets

Hypermultiplets exist in any dimension $d \leq 6$ and contain four real scalars. The corresponding geometries are quaternionic. We only consider the case where an action exists.¹ Then the admissible scalar geometries are Riemannian special holonomy geometries contained in Berger’s list. For rigid hypermultiplets the geometry is hyper-Kähler [44], which means that the holonomy group of the scalar manifold Q_{4n} must be contained in the compact form of the symplectic group

$$\text{Hol}(Q_{4n}) \subset USp(2n) \subset SO(4n) ,$$

where n is the number of hypermultiples, i.e. $4n$ is the real dimension of Q_{4n} . This is equivalent to the existence of three integrable and parallel complex structures I_i , $i = 1, 2, 3$, which satisfy

¹ If only the existence of supersymmetric equations of motion is required, the scalar manifold need not be equipped with a metric, and the admissible geometries are more general [43].

the quaternionic algebra

$$I_i \circ I_j = I_k, \quad i, j, k \text{ cyclic}, \quad (I_i)^2 = -\mathbb{1}, \quad (1)$$

and act isometrically. Therefore Q_{4n} is Kähler with respect to all three complex structures.

If hypermultiplets are coupled to supergravity, the scalar manifold \bar{Q} must be quaternion-Kähler [45], which means that the holonomy group must satisfy

$$\text{Hol}(\bar{Q}_{4n}) \subset SU(2) \cdot USp(2n) \subset SO(4).$$

It is understood that the $SU(2)$ factor is non-trivial so that hyper-Kähler is not a subcase. For hypermultiplets this is automatic because of the additional condition that the Ricci scalar satisfies [45]

$$R(\bar{Q}_{4n}) = -8n(n+1).$$

Thus supergravity hypermultiplet manifolds have a negative Ricci scalar, and are Einstein manifolds, but not Ricci flat, while rigid hypermultiplets manifolds are Ricci flat. Moreover quaternion-Kähler geometry is much more complicated than hyper-Kähler geometry, and arguably it is the richest, most complicated and least understood geometry on Berger's list. This is one of the main obstacles for mastering the non-perturbative corrections to hypermultiplets arising in string compactifications.

Despite their name, quaternion-Kähler manifolds are in general neither Kähler, nor even complex. The restriction of the holonomy group only implies the local existence of three almost complex structures satisfying the quaternionic algebra (1). These triplets can be patched together by $SU(2)$ transformations, but they neither need to be globally defined, nor to be integrable. The only global object that can be constructed is a four-form which is an $SU(2)$ singlet. In general one does neither have complex coordinates, nor a Kähler potential at one's disposal when working with quaternion-Kähler manifolds.

However, there is a very useful relation between quaternion-Kähler geometry and hyper-Kähler geometry: for each quaternion-Kähler manifold \bar{Q}_{4n} of real dimension $4n$ there exists the so-called Swann bundle or hyper-Kähler cone [46], a hyper-Kähler manifold Q_{4n+4} of real dimension $4n+4$, which admits a homothetic, tri-holomorphic action of the group \mathbb{H}^* of invertible quaternions. Conversely, every quaternion-Kähler manifold can be obtained by taking a 'superconformal quotient' of the associated hyper-Kähler cone:

$$\bar{Q}_{4n} = Q_{4n+4}/\mathbb{H}^*.$$

This construction naturally occurs in the superconformal formulation of hypermultiplets, which uses the gauge equivalence between $n+1$ superconformal hypermultiplets and n hypermultiplets coupled to Poincaré supergravity [47].² In this context, the action of

$$\mathbb{H}^* \simeq \mathbb{R}^{>0} \cdot SU(2)$$

corresponds to the action of dilatations and $SU(2)$ gauge transformations, which are part of the superconformal group, on the superconformal hypermultiplets. The gauge-equivalent theory of n hypermultiplets coupled to Poincaré supergravity is obtained by gauge-fixing these symmetries, which geometrically corresponds to taking a quotient with respect to \mathbb{H}^* .

We will see that superconformal vector multiplets and vector multiplets coupled to Poincaré supergravity are related by similar quotients, and that the scalar manifolds of the superconformal theories are always cones (or at least 'conical'). In the case of hypermultiplets, the cone has

² For concreteness, we are referring to four-dimensional hypermultiplets.

four extra dimensions. While the $SU(2)$ acts isometrically, the dilatations act homothetically, i.e. the metric of Q_{4n+4} is not invariant but rescaled by a specific constant factor. This gives the metric of Q_{4n+4} the structure of a Riemannian cone,

$$ds_Q^2 = dr^2 + r^2 ds_Y^2 ,$$

where the $(4n+3)$ -dimensional manifold Y is tri-Sasakian. $SU(2)$ acts isometrically on Y_{4n+3} and the quaternion-Kähler manifold \bar{Q}_{4n} is obtained from Y_{4n+3} by taking a quotient with respect to this action. We will compare this to the special geometry of vector multiplets in the following.

2.2. Four-dimensional vector multiplets

The geometries of four-dimensional vector multiplets are special Kähler geometries. They are in particular Kähler³, and there are various equivalent ways of defining what is ‘special’ about them.

2.2.1. Rigid supersymmetry The scalar manifolds of rigid four-dimensional vector multiplets are affine special Kähler. We will discuss three definitions.

The first definition, which is analogous to the original definition of projective special Kähler geometry [1], is in terms of special complex coordinates X^I , $I = 1, \dots, n$ on the scalar manifold N_{2n} , which correspond to the scalar components of vector supermultiplets [48, 19]. A Kähler manifold is affine special Kähler if the Kähler potential $K(X, \bar{X})$ can be obtained from a holomorphic function $F(X)$, called the prepotential, by the formula

$$K = -i(X^I \bar{F}_I - F_I \bar{X}^I) ,$$

where

$$F_I = \frac{\partial F}{\partial X^I} .$$

The origin of the existence of the additional special structure is electric-magnetic duality. Vector multiplets contain scalars X^I , fermions λ_i^I , $i = 1, 2$, and gauge fields A_μ^I with field strength $F_{\mu\nu}^I$. Electric-magnetic duality is the invariance of the field equations (not of the action) under symplectic transformations $\Omega \in Sp(2n, \mathbb{R})$, which act linearly on the vector $(F_{\mu\nu}^I, G_{I|\mu\nu})$ containing the field strength and dual field strength. The dual field strength are $G_{I|\mu\nu}^\pm \propto \partial \mathcal{L} / \partial F_{\mu\nu}^{I|\pm}$, where \mathcal{L} is the Lagrangian, and where ‘ \pm ’ denotes the projection onto the (anti-)selfdual part. Supersymmetry implies that (X^I, F_I) also transforms as a symplectic vector.

We note in passing that there are symplectic bases where the components F_I are not the gradient of a function [49]. This is not a problem because the definition can be reformulated in terms of the symplectic vector (X^I, F_I) . Moreover, one can always go to a frame where a prepotential exists by a symplectic transformation.

There are various ways of defining special Kähler geometry in a coordinate free way [17, 18, 19, 20, 21, 22]. The following definition of affine special Kähler geometry is intrinsic in the sense that it only uses the manifold Q_{2n} and the canonical bundles associated with it, i.e. the tangent and cotangent bundle and the resulting tensor bundles [20]: an affine special Kähler manifold is a Kähler manifold which is equipped with a ‘special’ connection ∇ which is (i) flat, (ii) torsion-free, (iii) symplectic (the Kähler form is parallel), and satisfies (iv)

$$d^\nabla I = 0 ,$$

³ Again, we are insisting on the existence of an action principle, which implies that the scalar manifold must carry a metric.

where I is the complex structure (which here is interpreted as a vector valued one-form). In local coordinates

$$\nabla_{[a} I_{c]}^b = 0 .$$

Thus the complex structure is not ∇ -parallel, but satisfies a weaker condition. Note that except in the trivial case of a flat metric, the connection ∇ is different from the Levi-Civita connection D , and that ∇ is not metric compatible.

For our purposes yet another definition is important, which provides a universal construction of affine special Kähler manifolds in terms of a model vector space [22]: an affine special Kähler manifold is a complex manifold of real dimension $2n$ which locally admits a holomorphic and Lagrangian immersion

$$\Phi : Q_{2n} \rightarrow T^*\mathbb{C}^n \simeq \mathbb{C}^{2n} .$$

Here $T^*\mathbb{C}^n$ is interpreted as a complex-symplectic vector space. The special Kähler data on Q_{2n} are induced by pulling back the corresponding standard data on $T^*\mathbb{C}^n$. This definition naturally connects to the one in terms of local coordinates. When using standard symplectic coordinates (X^I, W_I) on $T^*\mathbb{C}^n$, then the immersion satisfies

$$W_I = \frac{\partial F}{\partial X^I} ,$$

where the prepotential F is the generating function of the Lagrangian immersion $\Phi = dF$. This assumes that Q_{2n} is embedded as a graph, which is the generic situation. For non-generic immersions the function F might not exist, but one can always choose a different (generic) symplectic basis where it does.

2.2.2. Local supersymmetry In the local case there exists a similar variety of equivalent definitions of projective special Kähler geometry [1, 17, 18, 19, 20, 21, 22]. None of the existing definitions is intrinsic in the sense of the second definition of affine special Kähler geometry given in the previous section.

Like in affine special Kähler geometry, the Kähler potential $\mathcal{K}(z, \bar{z})$ of a projective special Kähler manifold Q_{2n} can be expressed in terms of a holomorphic prepotential $\mathcal{F}(z)$ when using special holomorphic coordinates z^i , $i = 1, \dots, n$ which correspond to the scalar components of vector supermultiplets:

$$\mathcal{K} = -\log(-i[(\mathcal{F} - \bar{\mathcal{F}}) - (z^i - \bar{z}^i)(\mathcal{F}_i + \bar{\mathcal{F}}_i)]) , \quad \mathcal{F}_i = \frac{\partial \mathcal{F}}{\partial z^i} . \quad (2)$$

This formula is usually not used as a definition, but derived as a result. The disadvantage of (2) is that symplectic covariance is not manifest. The reason is that the supergravity multiplet contains one additional vector field, the graviphoton. The total $n + 1$ gauge fields and their duals transform linearly under the symplectic group $Sp(2n + 2, \mathbb{R})$, but one cannot construct a symplectic vector out of the n scalars z^i . This is also a complication when constructing extremal black holes solutions, which work by ‘balancing’ scalars against vector fields.

There are various ways of making symplectic covariance manifest. We will use the original definition which arises in the context of the superconformal formalism [1], and uses the gauge equivalence between a theory of $n + 1$ superconformal vector multiplets and n vector multiples coupled to Poincaré supergravity. Rigid superconformal invariance requires that the prepotential is homogeneous of degree 2. Geometrically, this implies that the scalar manifold N_{2n+2} is a *conical* affine special Kähler manifold [20, 22], meaning that it admits a homothetic holomorphic action of

$$\mathbb{C}^* = \mathbb{R}^{>0} \cdot U(1) .$$

From the superconformal point of view $\mathbb{R}^{>0}$ are the dilatations contained in the superconformal group, which act as homotheties on N_{2n+2} , giving it the structure of a Riemannian cone,

$$ds_N^2 = dr^2 + r^2 ds_S^2 .$$

The basis S_{2n+1} of the cone is, by definition, a Sasakian manifold. It can be identified with the hypersurface selected by the D-gauge

$$e^{-\mathcal{K}} = -i(X^I \bar{F}_I - F_I \bar{X}^I) = 1 .$$

The $U(1)$ transformations are likewise part of the superconformal group and act isometrically, both on N_{2n+2} , and on S_{2n+1} . The associated theory of n vector multiplets coupled to Poincaré supergravity is obtained by first gauging the superconformal transformations and then gauge fixing the transformations not belonging to the Poincaré supergroup. The resulting scalar manifold \bar{N}_{2n} is obtained by taking a quotient with the respect to the \mathbb{C}^* action:

$$\bar{N}_{2n} \simeq N_{2n+2}/\mathbb{C}^* \simeq N_{2n+2}/U(1) . \quad (3)$$

As indicated this quotient can be interpreted as a symplectic quotient because S_{2n+1} is the level set of the moment map of the $U(1)$ isometry. \bar{N}_{2n} inherits a Kähler metric, thus the quotient is a Kähler quotient. Moreover \bar{N}_{2n} carries additional structure, and the above construction can be used as the definition of projective special Kähler geometry. The relation to the definition in terms of special coordinates is as follows: the homothetic \mathbb{C}^* -action implies that the holomorphic prepotential $F(X)$ of Q_{2n+2} is homogeneous of degree 2, which is the original definition of special Kähler geometry. Special coordinates z^i on \bar{Q}_{2n} are obtained from special coordinates X^I on Q_{2n+2} by setting $z^i = X^i/X^0$, and the prepotential $\mathcal{F}(z)$ used in (2) is related to $F(X)$ by

$$F(X^0, \dots, X^n) = (X^0)^2 F(1, X^1/X^0, \dots) = (X^0)^2 \mathcal{F}(z) .$$

It is straightforward to check that the metric induced on \bar{Q}_{2n} by the construction (3) is a Kähler metric with Kähler potential (2), by expressing

$$\mathcal{K} = -\log(-i(X^I \bar{F}_I - F_I \bar{X}^I))$$

in terms of the coordinates $X^0, z^i = \frac{X^i}{X^0}$. One then observes that this agrees with (2), up to a Kähler transformation, which removes the dependence on X^0 .

2.3. Five-dimensional vector multiplets

Having reviewed the quaternionic and complex versions of special geometry, we now turn the real case, to make some interesting observations. Real special geometry is the simplest of the special geometries, and this tends to obscure the analogy with the others. In the following we work out and stress the analogies. This does not only allow us to see the systematics, but will have important applications: the proof that the r -map preserves geodesic completeness, natural generalizations of special geometry, and the systematic construction of five-dimensional black holes by lifting four-dimensional instantons.

2.3.1. Rigid supersymmetry The scalar manifolds of rigid five-dimensional vector multiplets are characterized by two properties [50, 30]:

- (i) The scalar metric is Hessian, i.e. it can be obtained as the second derivative of a real function, the Hesse potential h :

$$g_{ij} = \partial_{i,j}^2 h .$$

(ii) The Hesse potential is a cubic polynomial, equivalently

$$\partial_k g_{ij} = \text{const} .$$

This definition assumes that we are using special coordinates σ^i , $i = 1, \dots, n$, which correspond to the scalar components of five-dimensional vector multiplets.

The geometrical (coordinate free) definition of a Hessian manifold is as follows [51]: a Hessian manifold is a (Pseudo-)Riemannian manifold (M, g) , equipped with a flat, torsion-free connection, such that the rank 3 tensor ∇g is totally symmetric. The special coordinates used in the first definition are the ∇ -affine coordinates for which $\nabla_i = \partial_i$. In terms of special coordinates, the total symmetry of ∇g implies the total symmetry of $\partial_i g_{jk}$ (and of the Christoffel symbols of first kind), which is the integrability condition for the existence of a Hesse potential.

The second condition follows from supersymmetry and gauge invariance. Supersymmetry implies the presence of a Chern-Simons term

$$\mathcal{L}_{CS} \propto C_{ijk} A^i \wedge F^j \wedge F^k ,$$

where $C_{ijk} \propto \partial_{i,j,k}^3 h$. Gauge invariance (up to boundary terms) requires that C_{ijk} are constant. We will refer to the scalar manifolds of rigid five-dimensional vector multiplets as affine special real manifolds, as part of our emphasis on the analogies to complex special geometry. The terminology more commonly used in the literature is ‘very special geometry’, or ‘very special real geometry’. Affine special real manifolds are special cases of Hessian manifolds, and it makes sense to view Hessian manifolds as a generalization of affine special real manifolds. For example, the rigid version of the r -map (obtained by dimensional reduction of five-dimensional vector multiplets) has a natural generalization to Hessian manifolds (corresponding to the dimensional reduction of non-supersymmetric theories of vector and scalar fields with couplings encoded by a Hesse potential) [51].

2.3.2. Local supersymmetry⁴ The coupling of five-dimensional vector multiplets and the underlying special geometry was constructed in [52]. The scalar manifolds \bar{M}_n of five-dimensional vector multiplets coupled to supergravity are not Hessian, but hypersurfaces in Hessian manifolds. One starts with a Hessian manifold M_{n+1} , with a Hesse potential h that is a *homogeneous* cubic polynomial. Then \bar{M}_n is defined as a level surface of the prepotential

$$\bar{M}_n \simeq \mathcal{H} = \{h = 1\} \subset M_{n+1} . \quad (4)$$

The natural metric on \bar{M}_n is the pull back of the Hessian metric $g_{(0)} = -\frac{1}{3}\partial^2 h$ on M_{n+1} to the hypersurface \bar{M}_n :

$$g_{\bar{M}} = i^* \left(-\frac{1}{3}\partial^2 h \right) . \quad (5)$$

Here i is the embedding $i : \bar{M}_n \rightarrow M_{n+1}$, and we will explain the prefactor $-\frac{1}{3}$ below. While given (4), this is the obvious choice, there are in fact infinitely many Hessian metrics on M_{n+1} which have the same pull back to \bar{M}_n . In particular, the formula given in [52] is, in our conventions,

$$g_{\bar{M}} = i^* \left(-\frac{1}{3}\partial^2 \log h \right) . \quad (6)$$

It is easy to see that (5), (6) define the same metric on \bar{M}_n because the log only changes the behaviour in the direction normal to the hypersurface. But as metrics on M_{n+1} $g_{(0)} = -\frac{1}{3}\partial^2 h$ and $g_{(1)} = -\frac{1}{3}\partial^2 \log h$ are different.

⁴ This section is based on [38] and work in progress by the first author and Vicente Cortés.

These observations and the contrast to the quaternionic and complex case raise the following questions. First, what are the properties of $g_{(0)}$ and $g_{(1)}$ and what singles them out among the Hessian metrics which have $g_{\bar{M}}$ as their pull back? Second, while \bar{M}_n has been defined as a hypersurface, could we also regard it as a quotient $\bar{M} = M/\mathbb{R}^{>0}$? Related to this, can we regard M as a cone over \bar{M} ?

To answer these questions we need to explore the properties of $g_{(0)}$ and $g_{(1)}$. We start with $g_{(0)}$ by observing that this metric is indefinite. Imposing that the induced metric $g_{\bar{M}}$ is positive definite, as required to have standard kinetic terms for the scalar fields, is easily seen to imply that $g_{(0)}$ has Lorentz signature. In our convention, which includes a minus sign in the definition, $g_{(0)}$ is ‘mostly plus’, with the negative direction corresponding to the direction normal to the hypersurface. The numerical factor $1/3$ has been introduced in order to comply with conventions used in the supergravity literature. Next we observe that $(M, g_{(0)})$ is not a Riemannian cone over $(\bar{M}, g_{\bar{M}})$. The metric $g_{(0)}$ has a homothetic Killing vector

$$\xi = \sigma^I \frac{\partial}{\partial \sigma^I} ,$$

where σ^I are special coordinates on M . Taking the Lie derivative one finds

$$L_\xi g_{(0)} = 3g_{(0)} ,$$

which means that the metric has weight 3 with respect to the transformation generated by ξ . Equivalently, the metric coefficients g_{IJ} have weight 1, and are homogeneous functions of degree 1 of the special coordinates. A Riemannian cone requires a homothety that satisfies

$$D\xi = \mathbb{1} \in \text{End}(TM) ,$$

where D is the Levi Civita connection. Equivalently,

$$D_X \xi = X$$

for all vector fields X . Decomposing this equation into its symmetric and antisymmetric part, one obtains

$$L_\xi g = 2g , \quad d(g^{-1}\xi) = 0 ,$$

where g is the metric and $g^{-1}\xi$ is the one form dual to the vector field ξ . In local coordinates:

$$D_i \xi_j + D_j \xi_i = 2g_{ij} , \quad \partial_i \xi_j - \partial_j \xi_i = 0 .$$

The first condition implies that the metric of a Riemannian cone must carry weight 2, rather than 3, under the homothety, while the second condition states that the homothetic Killing vector field must be hypersurface orthogonal.

We note that for the case at hand rescaling the homothetic Killing vector field is not an option, because we have the additional condition that the metric is Hessian. The homothety is the coordinate vector field associated with the special coordinates σ^I . This condition can be re-expressed as

$$\nabla \xi = \mathbb{1} ,$$

which fixes the normalization of ξ . The metric $g_{(0)}$ is not a Riemannian cone with respect to the homothety ξ but instead satisfies the similar condition

$$D\xi = \frac{3}{2}\mathbb{1} , \quad \nabla \xi = \mathbb{1} .$$

We will call Hessian manifolds with this property 3-conical. Replacing 3 in the above definition by an arbitrary number d , we obtain the definition of a d -conical Hessian manifold. The case $d = 2$ corresponds to a Riemannian cone in the usual sense. We remark that this definition can be adapted to Kähler manifolds. The conical affine special Kähler manifolds discussed before are Riemannian cones, i.e. 2-conical in the sense of the above definition. Thus the homothetic vector field ξ which generates dilatations on conical affine special Kähler manifolds satisfies $D\xi = \nabla\xi = \mathbb{1}$. The $U(1)$ Killing vector field is given by $I\xi$, where I is the complex structure.

The difference between five- and four-dimensional local vector multiplets can be understood from the superconformal perspective. The superconformal formulation of five dimensional supergravity has been worked out relatively recently [50, 43, 53]. The difference between five-dimensional and four-dimensional vector multiplets is that superconformal invariance requires that the four-dimensional prepotential is homogeneous of degree 2, while the five-dimensional Hesse potential is homogeneous of degree 3. As a result N is a cone over \bar{N} while M , equipped with the metric $g_{(0)}$, is 3-conical. From the superconformal point of view it is natural to regard $g_{\bar{M}}$ to arise from $g_{(0)}$ by dilatational gauge fixing. The direction normal to the hypersurface corresponds to the compensator field for dilatational symmetry, and such fields typically arise with a minus sign in front of their kinetic term.

We now turn to the metric $g_{(1)}$. With the chosen sign this metric is positive definite. This is necessary because the tensor field obtained by restricting $g_{(1)}$ to the hypersurface $h = 1$, is the gauge coupling matrix of the supergravity theory. Note that a five-dimensional supergravity theory with n vector multiplets has $n+1$ gauge fields, because the supergravity multiplet contains a gauge field, usually called the graviphoton. In the superconformal approach, one starts with a rigidly superconformal theory where both the scalar and vector kinetic terms contain the metric $g_{(0)}$. When gauging the superconformal symmetries and eliminating auxiliary fields, the gauge coupling matrix $g_{(0)}$ is replaced by $g_{(1)}$. In other words integrating out auxiliary fields effectively replaces the Hesse potential h by its logarithm $\log h$. The metric $g_{(1)}$ is again not a cone metric over $g_{\bar{M}}$. It is in fact something even simpler, namely the metric product of a one-dimensional factor and the metric $g_{\bar{M}}$. Introducing a coordinate r by

$$\xi = \frac{\partial}{\partial r} = \sigma^I \frac{\partial}{\partial \sigma^I} ,$$

the metric takes the form

$$g_{(1)} = dr^2 + g_{\bar{M}} .$$

For this metric the vector field ξ , which acts on r by translation and otherwise acts trivially, is not only a homothety but an isometry. Thus $g_{(1)}$ is homogeneous of degree 0 in the affine coordinates σ^I , and therefore the components $g_{(1)IJ}$ are homogeneous of degree -2 . We can obviously write \bar{M} as a quotient, $\bar{M} = M/\mathbb{R}$. To stress the analogy with the complex and quaternionic case, we can introduce the coordinate $\rho = e^r$, on which ξ acts by dilatations rather than translations, and then write $\bar{M} = M/\mathbb{R}^{>0}$.

3. The r -map

3.1. The rigid r -map

The dimensional reduction of a theory of five-dimensional vector multiplets induces a map between affine special real manifolds M_n and affine special Kähler manifolds N_{2n} , called the r -map:

$$r : M_n \mapsto N_{2n} .$$

Upon dimensional reduction the components A_5^I of the gauge fields along the reduced direction become scalars b^I , which are ‘axions’ in the sense that the five-dimensional gauge symmetry

induces an invariance under constant shifts. The resulting metric [30, 51, 33]

$$g_{IJ}(\sigma)d\sigma^I d\sigma^J \mapsto g_{IJ}(\sigma)(d\sigma^I d\sigma^J + db^I db^J) \quad (7)$$

is the natural metric on the tangent bundle of M_n , thus $N_{2n} \simeq TM_n$. $z^I = \sigma^I + ib^I$ are special coordinates on N_{2n} , and the prepotential of N_{2n} is related to the Hesse potential of M_n by

$$F(z^I) = h(\sigma^I + ib^I) .$$

The rigid r -map can be generalized. If g_{IJ} is any Hessian metric, then $g_{IJ}(\sigma)(d\sigma^I d\sigma^J + db^I db^J)$ is Kähler with Kähler potential

$$K(z, \bar{z}) = h(z + \bar{z})$$

and n commuting isometries acting by shifts. Conversely, any Kähler metric with n commuting shift isometries can be obtained from a Hessian metric by the generalized r -map [51].

3.2. The local r -map

When coupling vector multiplets to supergravity the effects of dimensional reduction are more complicated, because the reduction of the supergravity multiplet contributes additional degrees of freedom. The reduction of the metric gives a vector and a scalar, and the reduction of the graviphoton gives another scalar. The local r -map [25, 32, 33, 38] relates projective special real manifolds of dimension n to projective special Kähler manifolds of dimension $2n + 2$:

$$\bar{r} : \bar{M}_n \mapsto \bar{N}_{2n+2} .$$

Since the dimension does not simply double, it is clear that \bar{N}_{2n+2} is not the tangent bundle of \bar{M}_n . But based on the observations made above, it is nevertheless possible to uncover the underlying geometry. \bar{M}_n is a hypersurface given as a level set of the Hesse potential

$$h(h^0, \dots, h^n) = 1 .$$

When performing the dimensional reduction one can absorb the Kaluza-Klein scalar $\tilde{\sigma}$ into constrained scalars h^I , by setting

$$\sigma^I = e^{\tilde{\sigma}} h^I . \quad (8)$$

The scalars σ^I are unconstrained and can be interpreted as coordinates on the associated affine special real manifold M_{n+1} . The metric induced on M_{n+1} is the positive definite product metric $g_{(1)} = -\frac{1}{3}\partial^2 \log h$. Therefore the local r -map can be decomposed into two operations with a natural geometrical interpretation: the extension of \bar{M} to M , followed by the rigid r -map:

$$(\bar{M}_n, \bar{g}) \mapsto (M_{n+1}, g_{(1)}) \mapsto (N_{2n+2}, g_{(1)} \oplus g_{(1)}) ,$$

where $N_{2n+2} \simeq TM_{n+1}$. As discussed above, $(M_{n+1}, g_{(1)})$ is a metric product, and has an isometry generated by the Killing vector field $\xi = \sigma^I \partial_{\sigma^I}$. This extends to the Killing vector field

$$\xi = \sigma^I \frac{\partial}{\partial \sigma^I} + b^I \frac{\partial}{\partial b^I}$$

on N_{2n} , which combines with the n shift isometries into an $(n+1)$ -dimensional solvable Lie group \mathcal{L} . This group is the generic isometry group of projective special Kähler manifolds obtained from the local r -map. Generic means that it only contains the isometries generated by the r -map. If

the manifold \bar{M} has isometries, these will enlarge the isometry group of N . As a manifold, \bar{N} is locally the product of \bar{M} and the solvable Lie group \mathcal{L} :

$$N \simeq TM \simeq M \times \mathbb{R}^n \simeq \bar{M} \times \mathcal{L} .$$

This description of N is crucial for proving that the local r -map preserves completeness [38].

We remark that the local r -map can be generalized to the case where the Hesse potential is homogeneous of arbitrary degree p . While theories with $p \neq 3$ are not supersymmetric, the geometric structure governing their bosonic sector is very similar and all results stated above generalize in a straightforward way, because they only depend on homogeneity, but not on the degree [33, 38].

4. The c -map

We now turn to the c -map which is induced by the dimensional reduction of four-dimensional vector multiplets [23, 24]. Our presentation focuses on the role of the Hesse potential and of the special real coordinates of the four-dimensional theory, which allow us to present a new formulation of the c -map. In the rigid case our re-formulation makes it manifest that the scalar manifold of the three-dimensional is the cotangent bundle of the scalar manifold of the four-dimensional theory, equipped with its natural metric, while in the local case the metric is modified in a particular way. In the local case we can also show that the manifold of the three-dimensional theory is a group bundle, equipped with a bundle metric, which is crucial for proving that the local c -map preserves completeness.

Our approach is complementary to recent constructions of off-shell versions of the local c -map. The off-shell formulation of hypermultiplets requires infinitely many auxiliary fields. One way of dealing with this is to use projective superspace, which is the approach taken in [54]. The other approach is to use tensor multiplets [36]. In three dimensions, hypermultiplets and tensor multiplets are dual on-shell, but tensor multiplets admit an off-shell formulation with finitely many auxiliary fields. Both approaches allow to express the hypermultiplet geometry (specifically, the hyper-Kähler potential of the associated hyper-Kähler cone) in terms of the tensor multiplet prepotential.

4.1. The rigid c -map

The dimensional reduction of four-dimensional vector multiplets relates affine special Kähler manifolds N_{2n} and hyper-Kähler manifolds Q_{4n} . This defines the rigid c -map [23, 31]. Every four-dimensional gauge field gives rise to two scalars. The first scalar is the component of the four-dimensional gauge field along the reduced direction, the second scalar arises from dualizing the three-dimensional vector field. The resulting supermultiplets are rigid hypermultiplets, and so one obtains a map between affine special Kähler manifolds N_{2n} and hyper-Kähler manifolds Q_{4n} :

$$c : N_{2n} \mapsto Q_{4n} .$$

The hyper-Kähler manifold can be identified with the cotangent bundle of the special Kähler manifold,

$$Q_{4n} = T^*N_{2n} . \tag{9}$$

This generalizes the statement that the cotangent bundle of a Kähler manifold carries the structure of a hyper-Kähler manifold in a neighbourhood of the zero section [55, 56, 57, 58].

The relation (9) becomes manifest when we use special real coordinates $(q^a) = (\text{Re}(z^i), \text{Re}(F_i))$ instead of special holomorphic coordinates z^i on N_{2n} . The special real coordinates are affine coordinates with respect to the special connection ∇ , and are related to special holomorphic coordinates by a Legendre transformation. A projective special Kähler

metric is always also a Hessian metric, with the Hesse potential given by the Legendre transform of the imaginary part of the holomorphic prepotential:

$$H(q) = 2\text{Im}(F)(z(q)) - 2\text{Im}z^i(q)\text{Re}F_i .$$

The rigid c -map takes the form

$$g_{i\bar{j}}(z, \bar{z})dz^i d\bar{z}^j = H_{ab}(q)dq^a dq^b \mapsto H_{ab}(q)dq^a dq^b + H^{ab}(q)d\hat{q}_a d\hat{q}_b , \quad (10)$$

where $H_{ab} = \partial_{a,b}^2 H$ and where \hat{q}_a are the scalars arising from dimensionally reducing and dualizing the gauge fields. The hyper-Kähler structure on $Q_{4n} \simeq T^*N_{2n}$ is given canonically in terms of the special Kähler data on N_{2n} [23, 31]. Thus like the rigid r -map, the rigid c -map has a natural geometrical interpretation.

4.2. The local c -map

The local c -map is induced by dimensionally reducing supergravity with n vector multiplets, mapping them to $n + 1$ hypermultiplets [23, 24]. The four bosonic physical degrees of the four-dimensional gravity multiplet give rise to an additional hypermultiplet, usually called the universal hypermultiplet. Thus the local c -map relates projective special Kähler manifolds of dimension $2n$ to quaternionic Kähler manifolds of dimension $4n + 4$.

$$\bar{N}_{2n} \mapsto \bar{Q}_{4n+4} .$$

The explicit form for the metric is [24, 38]

$$g_{\bar{Q}} = g_{\bar{N}} + g_G ,$$

where

$$g_{\bar{N}} = g_{i\bar{j}}dz^i d\bar{z}^j$$

is the projective special Kähler metric on \bar{N} , and where

$$\begin{aligned} g_G = & \frac{1}{4\phi^2}d\phi^2 + \frac{1}{4\phi^2} \left(d\tilde{\phi} + (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I) \right)^2 + \frac{1}{2\phi} \mathcal{I}_{IJ}(p) d\zeta^I d\zeta^J \\ & + \frac{1}{2\phi} \mathcal{I}^{IJ}(p) (d\tilde{\zeta}_I + \mathcal{R}_{IK}(p)d\zeta^K) (d\tilde{\zeta}_J + \mathcal{R}_{JL}(p)d\zeta^L) . \end{aligned} \quad (11)$$

Here ϕ is the Kaluza-Klein scalar, $\tilde{\phi}$ the dualized Kaluza-Klein vector, the scalars ζ^I are the components of four-dimensional gauge fields along the reduced direction, and the scalars $\tilde{\zeta}_I$ are dual to the three-dimensional gauge fields. The couplings $\mathcal{R}_{IJ}(p)$ and $\mathcal{I}_{IJ}(p)$, which depend on $p \in \bar{N}$ (i.e. on z^i), are the real and imaginary part of the coupling matrix of the four-dimensional gauge fields,

$$\mathcal{N}_{IJ} = \mathcal{R}_{IJ} + i\mathcal{I}_{IJ} = \bar{F}_{IJ} + i \frac{N_{IK}z^K N_{LJ}z^L}{N_{MN}z^M z^N} , \quad N_{IJ} = 2\text{Im}F_{IJ} .$$

As for the local r -map, the geometrical interpretation is not immediately clear. The isometry group G of $g_{\bar{Q}}$ is a $(2n + 4)$ -dimensional Lie group. The metric $g_{\bar{Q}}$ has the structure of a ‘fibred product’, and it was shown in [38] that the fibres, parametrized by $\phi, \tilde{\phi}, \zeta^I, \tilde{\zeta}_I$ can be identified with the group G . Moreover, the fibres are equipped with a G -invariant metric, which depends smoothly on $p \in \bar{N}$. Thus \bar{Q} is a group bundle with local form

$$\bar{Q} \simeq \bar{N} \times G ,$$

and equipped with a bundle metric. One can make this explicit by rewriting g_G in terms of a left-invariant co-frame [38]. This rewriting of the metric is essential for proving that the local c -map preserves completeness.

Another interesting way of rewriting the local c -map is to use the Hesse potential (rather than the holomorphic prepotential) of the conical affine special Kähler manifold N associated to the projective special Kähler manifold \bar{N} . Already in [38] it was observed that the fibre metric g_G can be written as

$$g_G = \frac{1}{4\phi^2} d\phi^2 + \frac{1}{4\phi^2} (d\tilde{\phi} + \hat{q}_a \Omega^{ab} d\hat{q}_b)^2 + \frac{1}{2\phi} \hat{H}^{ab} d\hat{q}_a d\hat{q}_b .$$

Here $(\hat{q}_a) = (\tilde{\zeta}_I, \zeta^I)$,

$$\Omega = (\Omega_{ab}) = \begin{pmatrix} 0 & \mathbb{1}_{n+1} \\ -\mathbb{1}_{n+1} & 0 \end{pmatrix} , \quad \Omega^{-1} = (\Omega^{ab}) = \begin{pmatrix} 0 & -\mathbb{1}_{n+1} \\ +\mathbb{1}_{n+1} & 0 \end{pmatrix} ,$$

and

$$\hat{H} = (\hat{H}_{ab}) = \begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix} , \quad \hat{H}^{-1} = (\hat{H}^{ab}) = \begin{pmatrix} \mathcal{I}^{-1} & \mathcal{I}^{-1}\mathcal{R} \\ \mathcal{R}\mathcal{I}^{-1} & \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} \end{pmatrix} .$$

Note that \hat{q}_a , \hat{H}_{ab} and \hat{H}^{ab} transform linearly under symplectic transformations, whereas \mathcal{N}_{IJ} transforms fractionally linearly. In the above expression for g_G symplectic invariance is manifest.

More recently, this rewriting has been extended to the remaining variables $z^i, \phi, \tilde{\phi}$, resulting in an expression which is very similar to the ‘metric on the (co)tangent bundle form’ of the rigid c -map (10) [41]. This requires the following series of observations. First, the Hesse potential is associated to the conical affine special Kähler manifold N , and symplectic transformations act in a simple way on N , but not on \bar{N} . Therefore it is not possible to introduce special real coordinates on \bar{N} which transform linearly under the full symplectic group [42]. To circumvent this problem we re-express $g_{\bar{N}}$ in terms of quantities defined on N by

$$g_G = g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}} = g_{IJ} dX^I d\bar{X}^{\bar{J}} ,$$

where

$$g_{IJ} = \frac{\partial^2 \mathcal{K}}{\partial X^I \partial \bar{X}^{\bar{J}}} , \quad \mathcal{K} = -\log(-i(X^I \bar{F}_I - F_I \bar{X}^{\bar{I}})) ,$$

is a degenerate tensor field on N which by pull back gives the (non-degenerate) metric on \bar{N} . The expression $g_{IJ} dX^I d\bar{X}^{\bar{J}}$ formally depends on two additional degrees of freedom, corresponding to the radial direction of the cone N over the Sasakian S , and to the orbits of the $U(1)$ action on N . The first is eliminated by imposing the D-gauge⁵

$$-i(X^I \bar{F}_I - F_I \bar{X}^{\bar{I}}) = 1 .$$

Moreover $g_{IJ} dX^I d\bar{X}^{\bar{J}}$ is invariant under local $U(1)$ transformations, which eliminate the second additional degree of freedom.

The next step is to absorb the Kaluza-Klein scalar into the fields X^I living on N , as we did for the local r -map in (8). Defining⁶

$$Y^I = \phi^{1/2} X^I ,$$

⁵ We remark that this degree of freedom can be isolated and decoupled by suitable field redefinitions.

⁶ When comparing to [41], one has to replace $\phi \rightarrow e^\phi$.

the Kaluza-Klein scalar becomes a dependent field:

$$\phi = -N_{IJ}Y^I\bar{Y}^J = -i(Y^I\bar{F}_I - F_I\bar{Y}^I) .$$

This identifies the Kaluza-Klein scalar with the radial direction on N , which now has become a dynamical degree of freedom. Since the local $U(1)$ invariance is intact, the Y^I correspond to $2n + 2 - 1 = 2n + 1$ real scalars. Together with the dualized Kaluza-Klein vector $\tilde{\phi}$, which remains an independent field, we have $2n + 2$ real scalars, while the scalars \hat{q}_a obtained from the four-dimensional gauge fields add another $2n + 2$, bringing the count to $4n + 4$.

The next step is to replace the Y^I by the corresponding special real coordinates q^a . Like the Y^I , the q^a are subject to a local $U(1)$ transformation and therefore only correspond to $2n + 1$ independent real scalars. Using that the q^a are special real coordinates on N , one can show that

$$g_{IJ}dX^I d\bar{X}^J = \left(-\frac{1}{2H}H_{ab} + \frac{1}{4H^2}H_a H_b + \frac{1}{H^2}(\Omega_{ac}q^c \Omega_{bd}q^d) \right) dq^a dq^b ,$$

where H_a are the first derivatives of the Hesse potential. Next we remember that in the five-dimensional case the metric obtained after absorbing the Kaluza-Klein scalar could be expressed in terms of the logarithm of the Hesse potential of the rigid theory. This motivates us to define

$$\tilde{H} = -\frac{1}{2} \log H , \quad \tilde{H}_{ab} = \partial_{a,b}^2 \tilde{H} ,$$

and we find

$$g_{IJ}dX^I d\bar{X}^J = \left(\tilde{H}_{ab} - \frac{1}{4H^2}H_a H_b + \frac{1}{H^2}(\Omega_{ac}q^c \Omega_{bd}q^d) \right) dq^a dq^b .$$

It turns out that the term proportional to $H_a H_b$ now cancels against the term $(4\phi)^{-2}d\phi^2$ in g_G . To complete the rewriting, we remark that while the symplectic tensor \hat{H}_{ab} , which encodes the four-dimensional gauge couplings, is not a Hessian metric, it is related to the Hessian metric \tilde{H}_{ab} by

$$\tilde{H}_{ab} = \frac{1}{H}\hat{H}_{ab} - \frac{2}{H^2}(\Omega_{ac}q^c \Omega_{bd}q^d) .$$

This allows us to rewrite the metric $g_{\bar{Q}}$ such that the dependence on the underlying Hesse potential is exclusively through the Hessian metric \tilde{H}_{ab} , and H itself (which is proportional to the Kaluza-Klein scalar):⁷

$$g_{\bar{Q}} = \tilde{H}_{ab}(dq^a dq^b + d\bar{q}^a d\bar{q}^b) + \frac{1}{H^2}(q^a \Omega_{ab} dq^b)^2 + \frac{2}{H^2}(q^a \Omega_{ab} d\hat{q}^b)^2 + \frac{1}{4H^2}(d\tilde{\phi} + 2\hat{q}^a \Omega_{ab} d\hat{q}^b)^2 . \quad (12)$$

This form of the metric is very close indeed to the form of metrics obtained from the rigid c -map, with the couplings of the additional terms only depending on the constant matrix Ω and the Kaluza-Klein scalar.

5. Geodesic completeness

The local r -map allows to generate special Kähler manifolds from simpler special real manifolds, while the c -map generates quaternion-Kähler manifolds from special Kähler manifolds. By combining both maps one can start with a relatively simple special real manifold, encoded in a homogeneous cubic polynomial and construct an associated quaternion-Kähler manifold. This is very useful, because a complicated problem is related to a simpler one, both mathematically, describing quaternion-Kähler manifolds in terms of special real manifolds, and physically,

⁷ For convenience, we have replaced the coordinates \hat{q}_a by their duals \hat{q}^a , defined by $d\hat{q}^a = H^{ab}d\hat{q}_b$.

describing hypermultiplets in terms of vector multiplet data. One particularly interesting feature of this construction is that when starting with a non-homogeneous special real manifold, the result will be a non-homogeneous quaternion-Kähler manifold, at least generically.⁸ Thus the combined r - and c -map is a tool to construct non-homogeneous quaternion Kähler manifolds with all data encoded in a homogeneous cubic polynomial. One then naturally wonders about the properties of the manifolds produced in this way. Geodesic completeness is a very important geometrical property, which in physical terms means that no singularity in coupling space can be reached in finite time.

In the past, the symmetric and homogeneous spaces generated by the r -map and c -map have been studied exhaustively, including the classification of the homogeneous quaternion-Kähler spaces arising from the r and c -map [59, 25, 60]. Homogeneous spaces are geodesically complete, and it is known that the r -map and c -map relate homogeneous spaces to homogeneous spaces.

The improved understanding of the geometry of the local r - and c -map allows us to prove a very interesting statement which extends the classical results reviewed above [38]: the generalized⁹ local r -map and the local c -map preserve geodesic completeness. Thus complete special real manifolds can be used to construct complete, but generically non-homogeneous quaternion-Kähler manifolds. Moreover, the classification of complete special real manifolds appears to be tractable, at least in low dimensions [38].

The preservation of completeness follows from a general theorem, which states that given a complete Riemannian manifold, the Riemannian metrics on certain bundles are complete as well [38]. Specifically, given a complete Riemannian manifold (M_1, g_1) and a smooth family $g_2(p)$, $p \in M_1$ of G -invariant metrics on a homogeneous manifold $M_2 = G/K$, then the metric $g = g_1 + g_2$ on $M_1 \times M_2$ is complete, with isometric action of G . Moreover, this result generalizes from global products to bundles which take this form in a local trivialization. Since manifolds in the image of the local r -map and local c -map have the required form, it follows that both maps preserve completeness.

Let us indicate how the theorem is proved. Remember that there are two relevant concepts of completeness for Riemannian manifolds. Metric completeness means that all Cauchy series converge, geodesic completeness means that every geodesic ray can be extended to infinite length. The Hopf-Rinow theorem states that both conditions are equivalent to one another. One can show that completeness is also equivalent to the condition that every curve which is not contained in any compact subset has infinite length. Then the theorem is proved by estimating curve lengths on $M_1 \times M_2$ using that M_1 is assumed to be complete. For the problem of classification of complete special real manifolds we refer to [38] and future work.

While the mathematical merits of this result are obvious, the physical implications require further comment. If one was to consider supergravity as a fundamental theory, one would require that the coupling space has no singularities at finite distance, and therefore impose that the scalar manifold must be geodesically complete. However, the more likely candidate for a fundamental theory is string theory, with supergravity as a low-energy effective description. The scalar manifolds arising in $\mathcal{N} = 2$ supersymmetric string compactifications are not complete, but rather ‘incomplete in an interesting fashion.’ More precisely, one expects and indeed finds singularities at finite distance which corresponding to special loci in the moduli space where additional massless states occur. The most prominent example is the conifold singularity of Calabi-Yau threefolds [61].

Despite that string moduli spaces are not complete, we expect that our result is a significant step towards understanding the global geometry of string moduli spaces, in particular

⁸ There is no theorem forbidding that the result is homogeneous or symmetric, but the counting of the generic isometries generated by the r - and c -map indicates that such cases are exceptional.

⁹ Here we refer to the version of the local r -map where the degree of homogeneity of the Hesse potential is arbitrary.

hypermultiplet moduli spaces. The incorporation of perturbative and non-perturbative corrections to the r - and c -map, and the study of their role in the global geometry of string moduli spaces is an interesting topic for future work.

6. Generating stationary solutions

Dimensional reduction over time is a method which allows to construct stationary solutions. If all relevant fields in the reduced theory are scalars, the remaining problem of solving the scalar field equation is equivalent to constructing a harmonic map from space into the scalar manifold [62]. One standard approach is to find totally geodesic submanifolds and to find harmonic maps from space-time into them. An important subclass of solutions, which lifts to (supersymmetric as well as non-supersymmetric) extremal black holes, is provided by maps into submanifolds which are totally isotropic in addition to being totally geodesic.

In this section we will review how the temporal versions of the r -map and of the c -map can be used in the context of this construction [32, 33, 41]. In particular, we will see that a whole class of totally geodesic, totally isotropic submanifolds can be identified. Moreover, there is a set of canonical coordinates on these submanifolds which reduce the non-linear second order scalar field equations (corresponding to a harmonic map between Riemannian manifolds) to decoupled linear harmonic equations. We will also see that the five-dimensional and four-dimensional black hole attractor equations can be derived from Bogomol'nyi equations of the time-reduced theory.

Before starting, let us comment on the relation between our approach and others. Originally, the black hole attractor or stabilization equations were derived by imposing supersymmetry, i.e. the existence of Killing spinors. Imposing that the event horizon is finite implies enhanced supersymmetry on the horizon and forces the scalar fields to take prescribed fixed point values which are completely determined by the charges [7, 8, 9, 10]. The attractor equations determining the fixed point values can be formulated as a symplectically covariant equation relating two symplectic vectors, one containing the electric and magnetic charges, the other being proportional to the imaginary part of (X^I, F_I) , schematically

$$(\text{Im}X^I, \text{Im}F_I) \sim (p^I, q_I) . \quad (13)$$

This equation admits a generalization which specifies the black hole solutions globally (not only at the horizon) in terms of harmonic functions. As before there is an equation between two symplectic vectors, one containing the harmonic functions, the other is again proportional to the imaginary part of (X^I, F_I) . Schematically

$$(\text{Im}X^I, \text{Im}F_I) \sim (H^I, H_I) . \quad (14)$$

From these generalized attractor equations (also called generalized stabilization equations), the previous equations can be recovered by taking the near horizon limit. It was shown that the relations (14) are not only sufficient [12], but also necessary to obtain supersymmetric solutions [14]. Moreover, it is possible to include a class of higher-derivative terms [14].

It was already observed in [63] that the attractor equations could also be obtained from the equations of motion. Solving the scalar equations of motion can be re-formulated as a problem involving geodesic motion, and it is natural to combine this with dimensional reduction. Starting from [64, 65, 66] this approach has been used to construct non-supersymmetric extremal solutions. The supersymmetric attractor equations can be formulated as gradient flow equations, which are driven by the central charge [63, 67, 68]. For non-supersymmetric solutions it is also possible to reduce the second order field equations to first order gradient flow equations, which are then driven by a different function, often called a fake superpotential [69, 70, 71, 72]. The flow equation involve the physical scalars z^i rather than the symplectic vector (X^I, F_I) and are not manifestly symplectically covariant. For symmetric scalar target spaces, the construction of

both supersymmetric and non-supersymmetric extremal solutions can be related to integrable systems [73, 74, 75], [76, 77, 78].

Our approach is somewhat different in that while we also impose Bogomol'nyi equations, we do not derive gradient flow equations. Instead we solve the second order field equations directly in terms of harmonic functions and obtain the solution as an equation between two symplectic vectors, one containing the harmonic functions, the other encoding the scalars. Our method does not rely on non-generic isometries (as those needed to have a symmetric target space), but on the existence of a Hesse potential or holomorphic prepotential which encodes the geometry. Using the para-complex geometry of the target spaces obtained by dimensional reduction over time allows to identify totally geodesic, totally isotropic submanifolds which correspond to solutions. Special geometry also provides adapted coordinates which allow to reduce the non-linear second order equations to decoupled harmonic equations. Obtaining multicentered static extremal solution is as easy as obtaining single-centered static extremal solutions. The relation of our approach to the one based on gradient flow equations and integrability is not completely understood, but some aspects have been discussed in [33].

We remark that while we are focusing on using Euclidean solutions as a tool for generating black hole solutions, our Euclidean solutions are valid instanton solutions which are interesting in their own right [34]. This is an interesting topic in itself, see for example [79] for recent work on the classification of instantons in Einstein-Maxwell type theories.

In the above discussion we have referred to the four-dimensional version of the attractor mechanism for concreteness. There is also a five-dimensional version [80, 81, 82] which is accessible to our method. We will discuss this case first, as it is technically simpler.

6.1. Generating solutions from the local r -map

It is straightforward to generalize both the rigid and the local r -map such that one treats dimensional reduction over space, $\epsilon = -1$, and dimensional reduction over time, $\epsilon = 1$, in parallel:

$$g_{IJ}(\sigma)d\sigma^I d\sigma^J \rightarrow g_{IJ}(\sigma)(d\sigma^I d\sigma^J - \epsilon db^I db^J) , \quad (15)$$

where

$$g_{IJ} \sim -\partial_{I,J}^2 h$$

in the rigid case and

$$g_{IJ} \sim -\partial_{I,J}^2 \log h$$

in the local case. For time-like reductions the scalars coming from higher-dimensional gauge fields enter with the opposite sign. This implies that the scalar manifold N_{2n} has indefinite ('split') signature $(+)^n(-)^n$, and therefore has totally isotropic submanifolds of dimension n .

While for space-like reductions the scalars naturally combine into complex scalars $X^I = \sigma^I + ib^I$, for time-like reductions they combine into para-complex scalars

$$X^I = \sigma^I + eb^I ,$$

where $e^2 = 1$, $\bar{e} = -e$. Para-complex geometry is in many respects analogue to complex geometry, and one can view complex and para-complex geometry as different real forms of complex-Riemannian geometry. The concepts of para-Hermitian, para-Kähler, special para-Kähler, para-hyper-Kähler and para-quaternion-Kähler manifolds can be defined, and play a role in the Euclidean version of special geometry [30, 31, 32]. In this article we will not make heavy use of this formalism. In particular, we will not use para-holomorphic coordinates, but focus on suitable real coordinate systems instead.

At this point we can introduce another generalization. Supersymmetry requires that the Hesse potential has degree 3 (rigid case) or even is homogeneous of degree 3 (local case). However

the rigid r -map works for any Hessian manifold [51], and the local r -map can be generalized to the case where h is a homogeneous function of arbitrary degree p [33, 38]. We will refer to this class of manifolds as generalized special real geometry. For $p \neq 3$ there are no associated supersymmetric theories, but there is a corresponding class of non-supersymmetric theories of scalars, gauge fields and gravity, with the couplings controlled by the generalized special real geometry. The Lagrangian takes the form

$$\mathcal{L} \sim -\frac{1}{2}R_{(5)} - g_{IJ}\partial_\mu h^I \partial^\mu h^J - g_{IJ}F_{\mu\nu}^I F^{J|\mu\nu} + \dots ,$$

where it is understood that the scalars h^I are constrained to the hypersurface $h(h^I) = 1$, and where the omitted terms are not relevant for the solutions under consideration (i.e. the given terms must correspond to a consistent truncation).

We now review the use of the local r -map in constructing five-dimensional black hole solutions in the context of generalized special geometry, i.e. the Hesse potential h is homogeneous of arbitrary degree p . The relation between the five-dimensional and four-dimensional metric is:

$$ds_{(5)}^2 = -e^{2\tilde{\sigma}}(dt + \mathcal{A}_m dx^m)^2 + e^{-\tilde{\sigma}} ds_{(4)}^2 , \quad (16)$$

where $\tilde{\sigma}$ is the Kaluza-Klein scalar and where \mathcal{A}_m is the Kaluza-Klein vector. The decomposition has been chosen such that the gravitational part of the action remains in the canonical Einstein Hilbert form upon reduction. We will focus on static (non-rotating) solutions, characterized by a vanishing Kaluza-Klein vector. We will also impose that the four-dimensional Euclidean metric $ds_{(4)}^2$ is flat. We will refer to this class of solutions as extremal static solutions, because it includes extremal static black holes.

The reduced Lagrangian takes the form

$$\mathcal{L} \sim -\frac{1}{2}R_{(4)} - g_{IJ}(\partial_m \sigma^I \partial^m \sigma^J - \partial_m b^I \partial^m b^J) + \dots .$$

To solve the Einstein equations with a flat metric, we need to impose that the energy momentum tensor of the four-dimensional theory vanishes. This can be achieved by imposing the extremal instanton ansatz

$$\partial_m \sigma^I = \pm \partial_m b^I ,$$

which selects totally isotropic submanifolds of N . For $p = 3$, supersymmetric solutions correspond to taking the same sign for all pairs of fields, while different choices of signs correspond to non-supersymmetric extremal solutions. If the scalar metric g_{IJ} has discrete isometries,

$$g_{IJ}R^I{}_K R^J{}_L = g_{LM} ,$$

one can generalize the ansatz to

$$\partial_m \sigma^I = R^I{}_J \partial_m b^J .$$

This corresponds to ‘rotating the charges’ in the corresponding black hole solutions, which is a technique for obtaining non-supersymmetric solutions from supersymmetric solutions [69, 70].

It is clear that the totally isotropic submanifolds picked by the ansatz are totally geodesic and hence lead to harmonic maps into N , because we are choosing eigendirections of the para-complex structure. Since N is para-Kähler, the eigendistributions of the para-complex structure are not only integrable and totally geodesic, but parallel and flat with respect to the Levi-Civita connection [32, 33]. It is possible to verify this directly by using suitable local coordinates, as we will review now.

In a flat background, the scalar equations of motion are

$$\begin{aligned}\partial^m(g_{IJ}\partial_m\sigma^J) - \frac{1}{2}\partial_I g_{JK}(\partial_m\sigma^J\partial^m\sigma^K - \partial_m b^J\partial^m b^K) &= 0, \\ \partial^m(g_{IJ}\partial_m b^J) &= 0.\end{aligned}\tag{17}$$

After imposing the extremal instanton ansatz or its generalization, this reduces to

$$\partial^m(g_{IJ}\partial_m\sigma^J) = 0.$$

For the Hessian metric g_{IJ} we can define dual coordinates σ_I by

$$\partial_m\sigma_I = g_{IJ}\partial_m\sigma^J.$$

Note that since g_{IJ} is homogeneous of degree -2 , this implies $\sigma_I = -g_{IJ}\sigma^J$. In terms of dual coordinates, the scalar equations reduce to decoupled harmonic equations

$$\partial^m\partial_m\sigma_I = 0,$$

and the solution is given in terms of n harmonic functions H_I ,

$$\sigma_I = H_I.\tag{18}$$

The choice

$$H_I = h_I + \sum_{k=1}^N \frac{q_{I,k}}{|x - x_{(k)}|^2}$$

leads to multi-centered instanton solutions which lift to multi-centred extremal black hole solutions, with charges $q_{I,k}$ located at the centers $x_{(k)}$, which correspond to the positions of the event horizons. Expressing the solution (18) in terms of five-dimensional quantities, namely the Kaluza-Klein scalar $\tilde{\sigma}$ and the constrained scalars h^I , we recover the five-dimensional attractor equations [81, 82]

$$e^{-\tilde{\sigma}} \frac{\partial h}{\partial h^I} = H_I.$$

The line element of the black hole is determined by

$$e^{\tilde{\sigma}} = h(\sigma)^{1/p},$$

where p is the degree of the Hesse potential h . The ADM mass is given by

$$M_{ADM} = \frac{3}{2} \oint_{S_\infty^3} d^3\Sigma^m e^{-\tilde{\sigma}} \partial_m \tilde{\sigma},$$

and it can be shown that this agrees with the instanton action of the underlying four-dimensional Euclidean solution [33].¹⁰ Taking the near horizon limit of the attractor equations, one obtains the attractor equations

$$Z \left. \frac{\partial h}{\partial h^I} \right|_* = q_I,$$

where $*$ denotes evaluation on the horizon, and where

$$Z = (r^2 e^{-\tilde{\sigma}})_*.$$

¹⁰ There are some subtleties concerning the zero modes of the axionic scalars, which are discussed in [34].

These equations determine the horizon values h_*^I in terms of the charges q_I , which is the celebrated black hole attractor mechanism. For $p = 3$, Z agrees with the central charge of the supersymmetry algebra, up to normalization, $Z = \frac{1}{p} q_I h_*^I$.

For illustration let us give the explicit solution for the Hesse potential $h = \sigma^1 \dots \sigma^p$ [33]. For $p = 3$ this corresponds to the so-called STU-model. The line element is

$$ds_{(5)}^2 = -(H_1 \dots H_p)^{-2/p} dt^2 + (H_1 \dots H_p)^{1/p} \delta_{mn} dx^m dx^n ,$$

and the five-dimensional scalars

$$h^I = \left(\frac{\prod_{K \neq I} H_K}{H_I^{p-1}} \right)^{1/p}$$

have the limit

$$h^I \rightarrow \left(\frac{\prod_{K \neq I} q_{m,K}}{H_{m,I}^{p-1}} \right)^{1/p}$$

at the m -th center. The entropy of the m -th center is

$$\mathcal{S}_m = \frac{\pi^2}{2} Z_m^{3/2} = \frac{\pi^2}{2} \sqrt{q_{1,m} \dots q_{p,m}} .$$

6.2. Generating solutions from the local c -map

When adapting the above construction to the local c -map one encounters additional complications. This is partly due to the fact that four-dimensional black holes can carry both electric and magnetic charges. If one switches off the magnetic charges, and discards supersymmetry, then the construction reviewed in the previous section is straightforward to adapt to arbitrary dimensions, with only minor changes in the numerical values of coefficients (in particular the decomposition of the metric (16)).

However, the new form of the c -map based on the Hesse potential and special real coordinates allows to generalize the above construction to the case of the local c -map [41]. The question whether there is a generalized version of special Kähler geometry to which the construction can be extended remains open. The problem is that any such generalization will introduce an asymmetry (different scaling weights) between electric and magnetic degrees of freedom, and it is currently not clear how to include this into the formalism. Therefore we remain within the realm of special Kähler geometry proper.

Including time-like reductions in the previous calculations is straightforward and only leads to a few sign changes, which we parametrize by $\epsilon = -1$ for space-like and $\epsilon = 1$ for time-like reduction. After dimensional reduction one obtains the following three-dimensional Lagrangian for the bosonic fields [41]

$$\begin{aligned} \mathcal{L} \sim & -\frac{1}{2} R_3 - \tilde{H}_{ab} (\partial_m q^a \partial^m q^b - \epsilon \partial_m \hat{q}^a \partial_m \hat{q}^b) \\ & - \frac{1}{H^2} (q^a \Omega_{ab} \partial_m q^b)^2 + \epsilon \frac{2}{H^2} (q^a \Omega_{ab} \partial_m \hat{q}^b)^2 \\ & - \frac{1}{4H^2} (\partial_m \tilde{\phi} + 2 \hat{q}^a \Omega_{ab} \partial_m \hat{q}^b) . \end{aligned} \tag{19}$$

This is the Einstein-Hilbert term combined with a non-linear sigma model, and for $\epsilon = -1$ we recover the metric (12). We focus on solutions which will generate extremal static black holes

and impose that the three-dimensional metric is flat.¹¹ To solve the Einstein equations, the energy momentum tensor must vanish identically, which selects totally isotropic submanifolds of \bar{Q} . For $\epsilon = 1$ it is clear on general grounds that \bar{Q} is para-Quaternion-Kähler [30, 31], but we will not use this directly. Rather we will describe how the equations of motion can be solved explicitly in terms of harmonic functions, using dual coordinates and suitable ansätze. Since the scalar Lagrangian is a combination of perfect squares, a standard strategy is to impose the Bogomol'nyi equations resulting from the vanishing of these squares.

In particular, the term $\tilde{H}_{ab}(\partial_m q^a \partial^m q^b - \partial_m \hat{q}^a \partial_m \hat{q}^b)$ in the first line (setting $\epsilon = 1$) is analogous to the case of the local r -map. This motivates us to impose the extremal instanton ansatz

$$\partial_m q^a = \pm \partial_m \hat{q}^a ,$$

and to introduce dual coordinates defined by

$$\partial_m q_a = \tilde{H}_{ab} \partial_m q^b .$$

Using that \tilde{H}_{ab} is homogeneous of degree -2 , and various other previous results, we obtain

$$q_a = \tilde{H}_a = -\tilde{H}_{ab} q^b = \frac{1}{H}(-v_I, u^I) ,$$

where

$$u^I \simeq \text{Im} X^I , \quad v_I \simeq \text{Im} F_I$$

are proportional the standard dual special real coordinates. More precisely, affine special Kähler manifolds do not only admit one special connection, but a whole S^1 -family thereof, which is generated by the action of the complex structure [20, 22, 32]. Whereas $\text{Re} X^I$, $\text{Re} F_I$ are affine coordinates for the original special connection, $\text{Im} X^I$ and $\text{Im} F_I$ are affine coordinates for the ‘opposite’ family member with parameter value $\theta = \pi$, if we parametrize the S^1 family by an angle $0 \leq \theta < 2\pi$. The variables q_a and u^I, v_I used above are related to the dual special real coordinates $\text{Im} X^I, \text{Im} F_I$ through rescaling by specific powers of H , which itself is proportional to the Kaluza-Klein scalar ϕ [41].

Compared to the previous section, we have additional terms in the second and third line of (19). These terms are not independent once we impose the extremal instanton ansatz. Due to the relative factor 2, the terms in the second line do not cancel but combine into a single terms. The same term appears within the perfect square in the third line. Now we remark that the term in the third line is proportional to the square of the field strength of the Kaluza-Klein vector. Thus imposing that the third line vanishes implies that the resulting solution is static (non-rotating). It is possible to obtain more general, rotating solutions, by only requiring that the sum of second and third line vanishes. We refer to [41] for this case and focus on static solutions, where the second and third line vanish separately.

It is then straightforward, though somewhat tedious to verify that the scalar equations reduce to

$$\partial^m \partial_m q_a = 0 ,$$

while all other equations are satisfied identically. Thus, as before, the solution is given in terms of harmonic functions.

By rewriting the equations

$$q_a = H_a ,$$

¹¹ Note that in three dimensions Ricci flatness already implies flatness. In higher dimensions we could generalize our construction to allow a Ricci flat metric on the reduced space. This is left to future work.

where H_a are $2n + 2$ harmonic functions, in terms of the four-dimensional variables, we recover the four-dimensional black hole attractor equations¹²

$$\phi^{-1/2}(X^I - \bar{X}^I) = iH^I, \quad \phi^{-1/2}(F_I - \bar{F}_I) = iH_I,$$

in a symplectically covariant form.

Moreover when rewriting the extremal instanton ansatz

$$\partial_m q^a = \pm \partial_m \hat{q}^a$$

in terms of four-dimensional quantities we obtain

$$\partial_m(\phi^{1/2}(X^I + \bar{X}^I)) = \mp(F_{0m}^{I|+} + F_{0m}^{I|-}), \quad \partial_m(\phi^{1/2}(F_I + \bar{F}_I)) = \mp(G_{I|0m}^+ + G_{I|0m}^-),$$

which shows that the real part of the symplectic vector (X^I, F_I) is proportional to the electric and magnetic potentials. For supersymmetric solutions this follows from the gaugino variation [12, 14], while here we obtain it as the the Bogomol'nyi equation associated to the first line of (19).

Not surprisingly our formalism shows many similarities with the superconformal approach and its emphasis on symplectic covariance. Since we work with the bosonic field equations rather than with Killing spinor equations, we can also obtain non-supersymmetric extremal solutions. Supersymmetric solutions have the same sign for all fields in the instanton ansatz. Non-supersymmetric solutions correspond to the generalized ansatz,

$$\partial_m q^a = R^a_b \partial_m \hat{q}^b$$

where the matrix R^a_b is a discrete isometry of \tilde{H}_{ab} . We refer to [35, 41] for results on non-supersymmetric, non-extremal and rotating solutions. Note that non-extremal black hole solutions have also been discussed recently in [83, 84].

Acknowledgments

This article is based on a talk given by T.M. at the QTS7 in Prague. He thanks the organizers of QTS7 for the invitation and for organising this inspiring conference. The work of T.M. is supported in part by STFC grants ST/G00062X/1 and ST/J000493/1. The work of O.V. is supported by an STFC studentship and by DAAD. Part of the material included in this article is based on unpublished work by V. Cortés and the first author. Both authors thank V. Cortés for useful discussions and ongoing collaboration on the topics reviewed in this article. T.M. also thanks B. de Wit, J. Louis, P. Smyth and S. Vandoren for useful discussions.

- [1] de Wit B and Van Proeyen A 1984 *Nucl. Phys.* **B245** 89
- [2] Seiberg N and Witten E 1994 *Nucl. Phys.* **B426** 19–52 (*Preprint hep-th/9407087*)
- [3] Seiberg N and Witten E 1994 *Nucl. Phys.* **B431** 484–550 (*Preprint hep-th/9408099*)
- [4] Kachru S and Vafa C 1995 *Nucl. Phys.* **B450** 69–89 (*Preprint hep-th/9505105*)
- [5] Ferrara S, Harvey J A, Strominger A and Vafa C 1995 *Phys. Lett.* **B361** 59–65 (*Preprint hep-th/9505162*)
- [6] Kachru S, Klemm A, Lerche W, Mayr P and Vafa C 1996 *Nucl. Phys.* **B459** 537–558 (*Preprint hep-th/9508155*)
- [7] Ferrara S, Kallosh R and Strominger A 1995 *Phys. Rev.* **D52** 5412–5416 (*Preprint hep-th/9508072*)
- [8] Strominger A 1996 *Phys. Lett.* **B383** 39–43 (*Preprint hep-th/9602111*)
- [9] Ferrara S and Kallosh R 1996 *Phys. Rev.* **D54** 1514–1524 (*Preprint hep-th/9602136*)
- [10] Ferrara S and Kallosh R 1996 *Phys. Rev.* **D54** 1525–1534 (*Preprint hep-th/9603090*)

¹² When comparing to [14], note that $e^{2f} = \phi^{-1}$, where f is the function used to parametrize the four-dimensional metric in [14]. Also note that the variables called Y^I in [14] differ from the Y^I used in this article by a factor ϕ . Also, when comparing to [41], remember to replace $\phi \rightarrow e^\phi$.

- [11] Behrndt K *et al.* 1997 *Nucl. Phys.* **B488** 236–260 (*Preprint hep-th/9610105*)
- [12] Behrndt K, Lust D and Sabra W A 1998 *Nucl. Phys.* **B510** 264–288 (*Preprint hep-th/9705169*)
- [13] Lopes Cardoso G, de Wit B and Mohaupt T 1999 *Phys.Lett.* **B451** 309–316 (*Preprint hep-th/9812082*)
- [14] Lopes Cardoso G, de Wit B, Kappeli J and Mohaupt T 2000 *JHEP* **12** 019 (*Preprint hep-th/0009234*)
- [15] Strominger A and Vafa C 1996 *Phys. Lett.* **B379** 99–104 (*Preprint hep-th/9601029*)
- [16] Maldacena J M, Strominger A and Witten E 1997 *JHEP* **9712** 002 (*Preprint hep-th/9711053*)
- [17] Strominger A 1990 *Commun. Math. Phys.* **133** 163–180
- [18] Castellani L, D’Auria R and Ferrara S 1990 *Class. Quant. Grav.* **7** 1767–1790
- [19] Craps B, Roose F, Troost W and Van Proeyen A 1997 *Nucl. Phys.* **B503** 565–613 (*Preprint hep-th/9703082*)
- [20] Freed D S 1999 *Commun. Math. Phys.* **203** 31–52 (*Preprint hep-th/9712042*)
- [21] Hitchin N 1999 *ArXiv Mathematics e-prints* (*Preprint arXiv:math/9901069*)
- [22] Alekseevsky D V, Cortes V and Devchand C 2002 *J. Geom. Phys.* **42** 85–105 (*Preprint math/9910091*)
- [23] Cecotti S, Ferrara S and Girardello L 1989 *Int. J. Mod. Phys.* **A4** 2475
- [24] Ferrara S and Sabharwal S 1990 *Nucl. Phys.* **B332** 317
- [25] de Wit B and Van Proeyen A 1992 *Commun. Math. Phys.* **149** 307–334 (*Preprint hep-th/9112027*)
- [26] de Wit B and Van Proeyen A 1995 Isometries of special manifolds (*Preprint hep-th/9505097*)
- [27] Robles-Llana D, Saueressig F and Vandoren S 2006 *JHEP* **0603** 081 (*Preprint hep-th/0602164*)
- [28] Alexandrov S, Pioline B, Saueressig F and Vandoren S 2009 *JHEP* **0903** 044 (*Preprint 0812.4219*)
- [29] Robles-Llana D, Rocek M, Saueressig F, Theis U and Vandoren S 2007 *Phys.Rev.Lett.* **98** 211602 (*Preprint hep-th/0612027*)
- [30] Cortes V, Mayer C, Mohaupt T and Saueressig F 2004 *JHEP* **03** 028 (*Preprint hep-th/0312001*)
- [31] Cortes V, Mayer C, Mohaupt T and Saueressig F 2005 *JHEP* **06** 025 (*Preprint hep-th/0503094*)
- [32] Cortes V and Mohaupt T 2009 *JHEP* **07** 066 (*Preprint 0905.2844*)
- [33] Mohaupt T and Waite K 2009 *JHEP* **10** 058 (*Preprint 0906.3451*)
- [34] Mohaupt T and Waite K 2011 *J. Phys.* **A44** 175403 (*Preprint 1011.6301*)
- [35] Mohaupt T and Vaughan O 2010 *Class. Quant. Grav.* **27** 235008 (*Preprint 1006.3439*)
- [36] de Wit B and Saueressig F 2006 *JHEP* **09** 062 (*Preprint hep-th/0606148*)
- [37] De Jaegher J, de Wit B, Kleijn B and Vandoren S 1998 *Nucl. Phys.* **B514** 553–582 (*Preprint hep-th/9707262*)
- [38] Cortes V, Han X and Mohaupt T 2011 Completeness in supergravity constructions accepted for publication in *Commun. Math. Phys.* (*Preprint 1101.5103*)
- [39] Lopes Cardoso G, de Wit B, Kappeli J and Mohaupt T 2006 *JHEP* **0603** 074 (*Preprint hep-th/0601108*)
- [40] Cardoso G L, de Wit B and Mahapatra S 2010 *JHEP* **06** 052 (*Preprint 1003.1970*)
- [41] Mohaupt T and Vaughan O The hesse potential, the c-map and black hole solutions to appear
- [42] Ferrara S and Macia O 2006 *Phys. Lett.* **B637** 102–106 (*Preprint hep-th/0603111*)
- [43] Bergshoeff E, Cucu S, Gheerardyn J, Halbersma R, Vandoren S and Van Proeyen A 2002 *JHEP* **10** 045 (*Preprint hep-th/0205230*)
- [44] Alvarez-Gaume L and Freedman D Z 1980 *Phys.Rev.* **D22** 846 revised version
- [45] Bagger J and Witten E 1983 *Nucl. Phys.* **B222** 1
- [46] Swann A 1991 *Math. Ann.* **289** 421–450
- [47] de Wit B, Kleijn B and Vandoren S 2000 *Nucl. Phys.* **B568** 475–502 (*Preprint hep-th/9909228*)
- [48] Ceresole A, D’Auria R and Ferrara S 1994 *Phys.Lett.* **B339** 71–76 (*Preprint hep-th/9408036*)
- [49] Ceresole A, D’Auria R, Ferrara S and Van Proeyen A 1994 (*Preprint hep-th/9412200*)
- [50] Bergshoeff E *et al.* 2001 *JHEP* **06** 051 (*Preprint hep-th/0104113*)
- [51] Alekseevsky D V and Cortés V 2009 *Commun. Math. Phys.* **291** 579–590 (*Preprint 0811.1658*)
- [52] Gunaydin M, Sierra G and Townsend P K 1984 *Nucl. Phys.* **B242** 244
- [53] Bergshoeff E *et al.* 2004 *Class. Quant. Grav.* **21** 3015–3042 (*Preprint hep-th/0403045*)
- [54] Rocek M, Vafa C and Vandoren S 2006 *JHEP* **02** 062 (*Preprint hep-th/0512206*)
- [55] Calabi E 1979 *Ann. Sci. École Norm. Sup. (4)* **12** 269–294 ISSN 0012-9593
- [56] Kaledin D 1997 *eprint arXiv:alg-geom/9710026* pp 10026–+
- [57] Gates S James J and Kuzenko S M 1999 *Nucl. Phys.* **B543** 122–140 (*Preprint hep-th/9810137*)
- [58] Gates S James J and Kuzenko S M 2000 *Fortsch. Phys.* **48** 115–118 (*Preprint hep-th/9903013*)
- [59] Alekseevskii D V 1975 *Izv. Akad. Nauk SSSR Ser. Mat.* **39** 315–362, 472 ISSN 0373-2436
- [60] Cortés V 1996 *Differential Geom. Appl.* **6** 129–168 ISSN 0926-2245 URL [http://dx.doi.org/10.1016/0926-2245\(96\)89146-7](http://dx.doi.org/10.1016/0926-2245(96)89146-7)
- [61] Strominger A 1995 *Nucl.Phys.* **B451** 96–108 (*Preprint hep-th/9504090*)
- [62] Breitenlohner P, Maison D and Gibbons G W 1988 *Commun. Math. Phys.* **120** 295
- [63] Ferrara S, Gibbons G W and Kallosh R 1997 *Nucl. Phys.* **B500** 75–93 (*Preprint hep-th/9702103*)
- [64] Goldstein K, Iizuka N, Jena R P and Trivedi S P 2005 *Phys. Rev.* **D72** 124021 (*Preprint hep-th/0507096*)
- [65] Tripathy P K and Trivedi S P 2006 *JHEP* **03** 022 (*Preprint hep-th/0511117*)

- [66] Kallosh R 2005 *JHEP* **0512** 022 (*Preprint* hep-th/0510024)
- [67] Moore G W 1998 (*Preprint* hep-th/9807087)
- [68] Denef F 2000 *JHEP* **08** 050 (*Preprint* hep-th/0005049)
- [69] Ceresole A and Dall'Agata G 2007 *JHEP* **03** 110 (*Preprint* hep-th/0702088)
- [70] Lopes Cardoso G, Ceresole A, Dall'Agata G, Oberreuter J M and Perz J 2007 *JHEP* **10** 063 (*Preprint* 0706.3373)
- [71] Perz J, Smyth P, Van Riet T and Vercknocke B 2009 *JHEP* **03** 150 (*Preprint* 0810.1528)
- [72] Ortin T 2011 *Phys. Lett.* **B700** 261–264 (*Preprint* 1103.2738)
- [73] Gunaydin M, Neitzke A, Pioline B and Waldron A 2006 *Phys. Rev.* **D73** 084019 (*Preprint* hep-th/0512296)
- [74] Gunaydin M, Neitzke A, Pioline B and Waldron A 2007 *JHEP* **09** 056 (*Preprint* arXiv:0707.0267[hep-th])
- [75] Bossard G, Michel Y and Pioline B 2010 *JHEP* **01** 038 (*Preprint* 0908.1742)
- [76] Chemissany W, Rosseel J, Trigiante M and Van Riet T 2010 *Nucl.Phys.* **B830** 391–413 (*Preprint* 0903.2777)
- [77] Andrianopoli L, D'Auria R, Orazi E and Trigiante M 2009 (*Preprint* 0905.3938)
- [78] Chemissany W *et al.* 2010 (*Preprint* 1007.3209)
- [79] Azreg-Ainou M, Clement G and Gal'tsov D V 2011 * Temporary entry * (*Preprint* 1107.5746)
- [80] Chamseddine A H, Ferrara S, Gibbons G W and Kallosh R 1997 *Phys. Rev.* **D55** 3647–3653 (*Preprint* hep-th/9610155)
- [81] Chamseddine A H and Sabra W A 1998 *Phys. Lett.* **B426** 36–42 (*Preprint* hep-th/9801161)
- [82] Chamseddine A H and Sabra W A 1999 *Phys. Lett.* **B460** 63–70 (*Preprint* hep-th/9903046)
- [83] Galli P, Ortin T, Perz J and Shahbazi C S 2011 *JHEP* **07** 041 (*Preprint* 1105.3311)
- [84] Meessen P and Ortin T 2011 (*Preprint* 1107.5454)